ME 141
Engineering Mechanics

Lecture 8: Moment of Inertia

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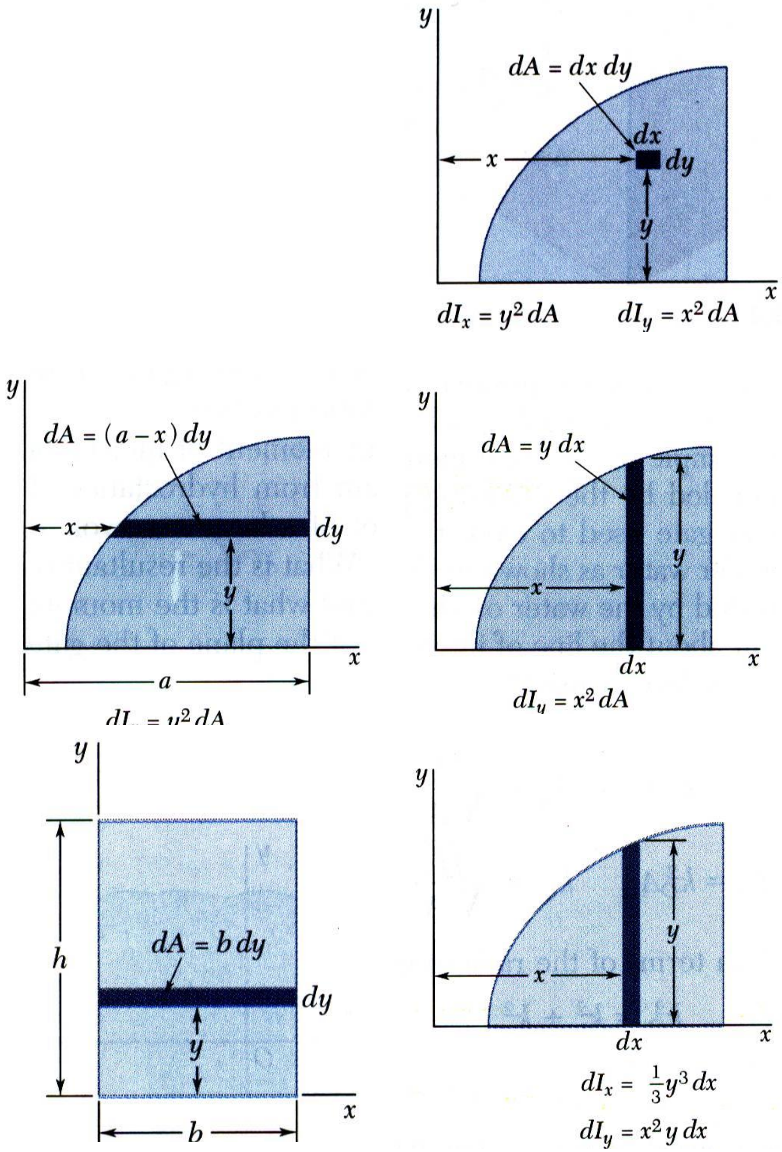
Courtesy: Vector Mechanics for Engineers, Beer and Johnston

Moment of Inertia of an Area



- *Second moments* or *moments of inertia* of an area with respect to the *x* and *y* axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

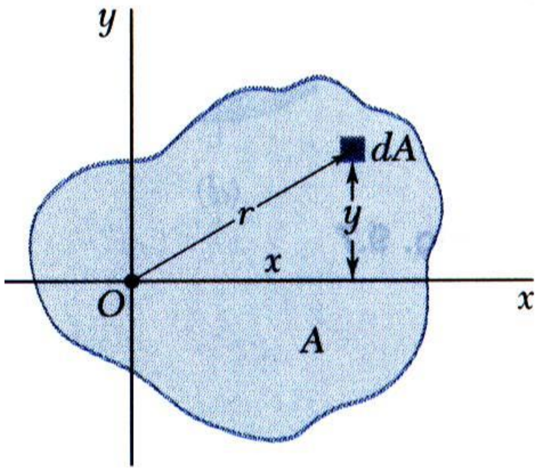
- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b dy = \tfrac{1}{3} bh^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\, dx$$

Polar Moment of Inertia



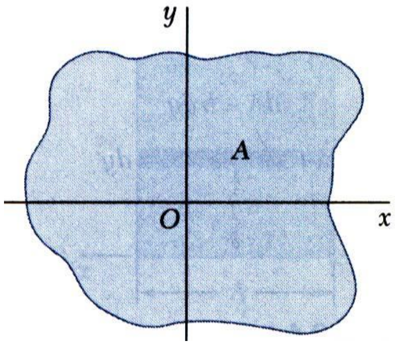
- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

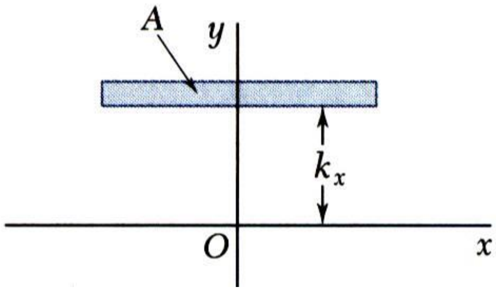
$$J_0 = \int r^2 dA$$

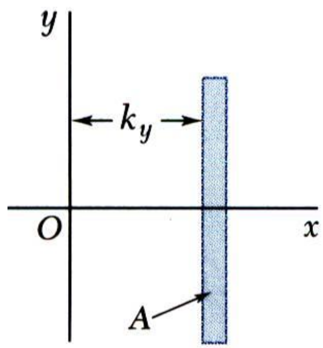
- The polar moment of inertia is related to the rectangular moments of inertia,

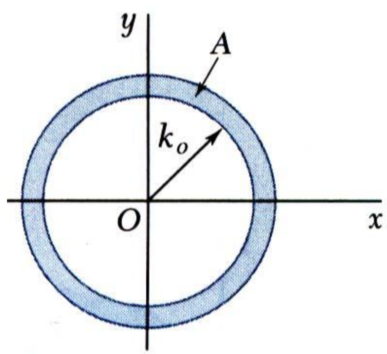
$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Radius of Gyration of an Area









- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

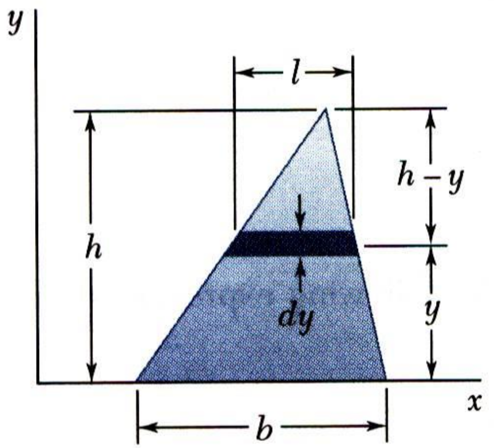
- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$

Sample Problem 9.1



Determine the moment of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

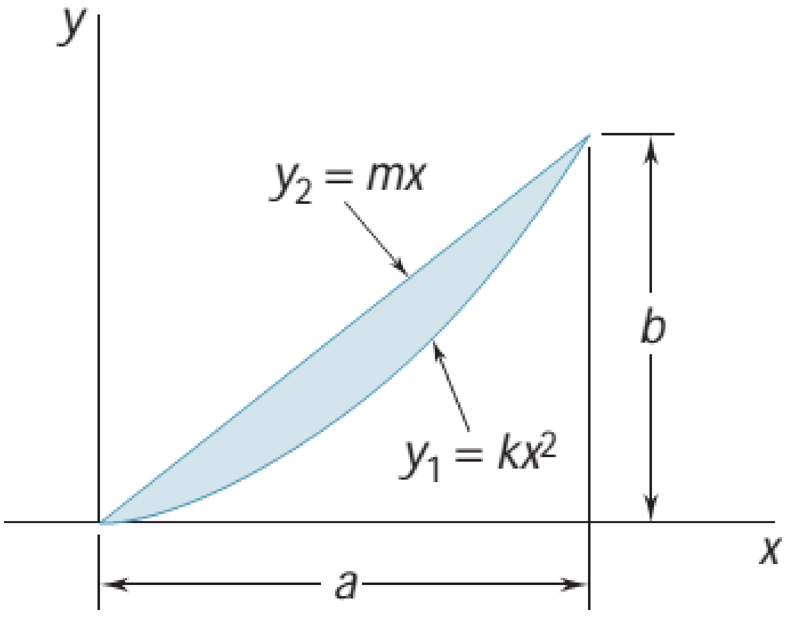
$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

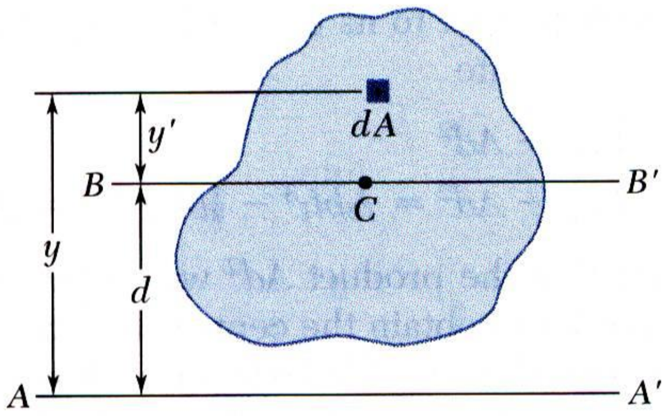
$$I_x = \frac{bh^3}{12}$$

Prob # 9.9 and 9.12

Determine by direct integration the moment of inertia of the shaded area with respect to the x and y axes.



Parallel Axis Theorem



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

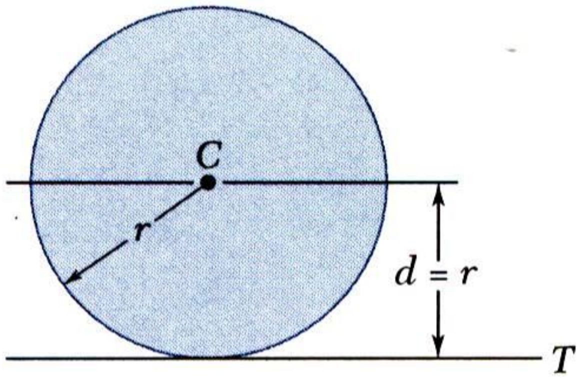
$$I = \int y^2 dA$$

- The axis *BB'* passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

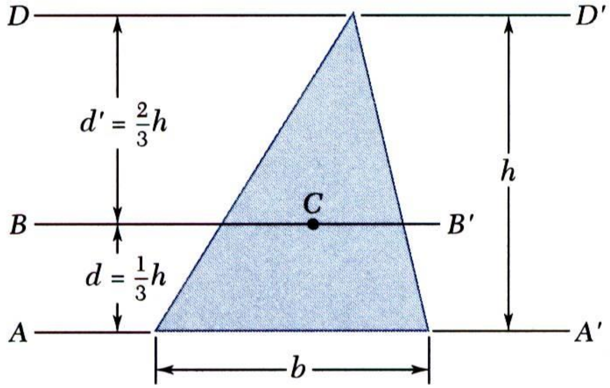
$$\boxed{I = \bar{I} + Ad^2}$$ *parallel axis theorem*

Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$

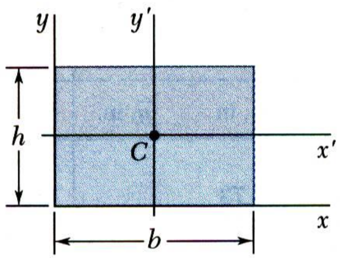
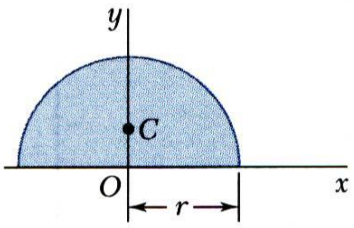
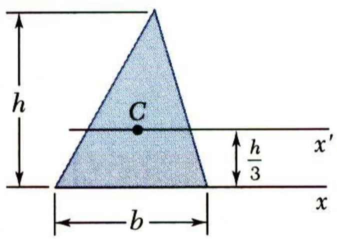
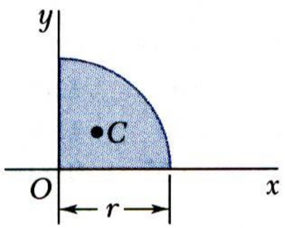
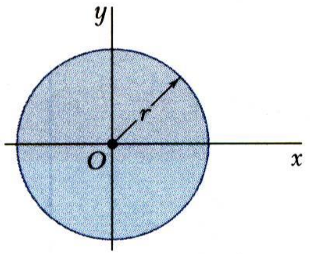
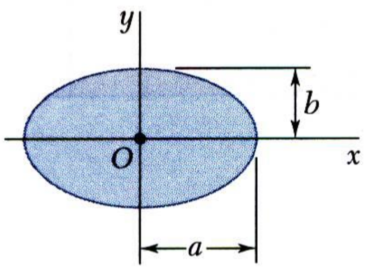


- Moment of inertia of a triangle with respect to a centroidal axis,

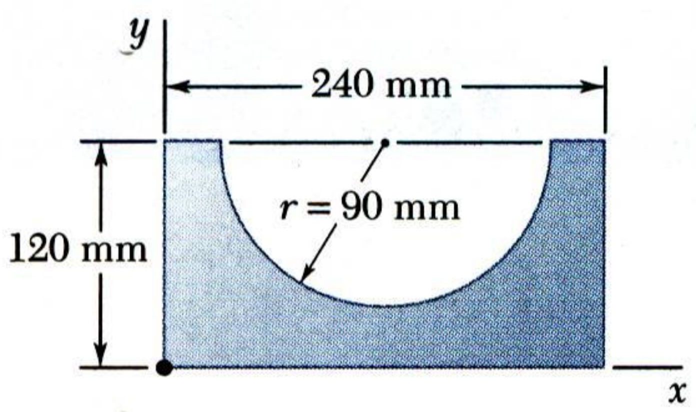
$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

Moments of Inertia of Composite Areas

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas $A_1, A_2, A_3, \ldots$, with respect to the same axis.

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Sample Problem 9.5

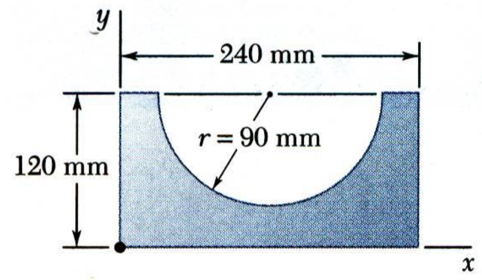


Determine the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

Sample Problem 9.5

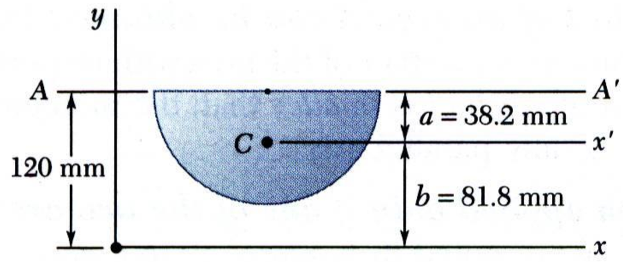


<u>SOLUTION</u>:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2\times10^6\,\text{mm}^4$$



Half-circle:

moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\text{mm}^4$$

moment of inertia with respect to *x'*,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = \left(25.76\times10^6\right)\left(12.72\times10^3\right)$$
$$= 7.20\times10^6\,\text{mm}^4$$

$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\text{ mm}$$

$$\text{b} = 120 \text{ - a} = 81.8\text{ mm}$$

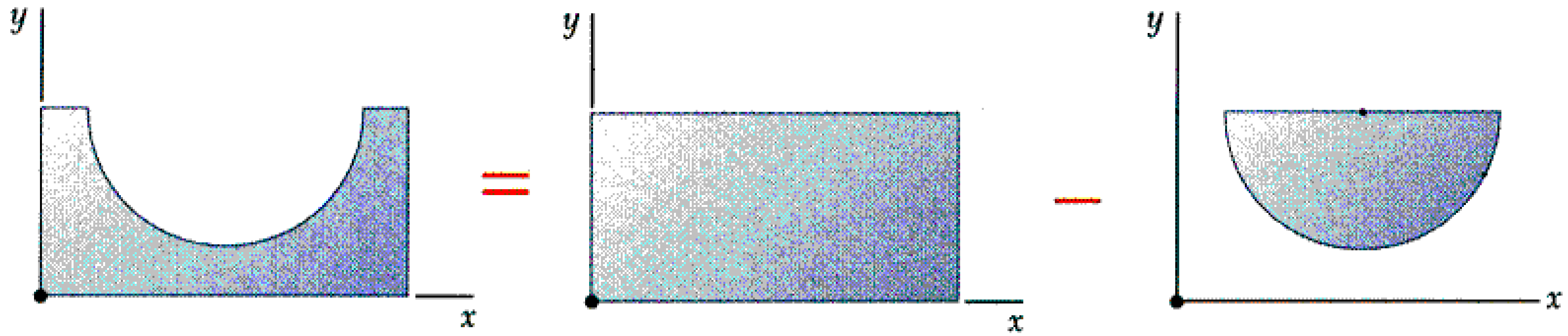
$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$
$$= 12.72\times10^3\,\text{mm}^2$$

moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + \left(12.72\times10^3\right)(81.8)^2$$
$$= 92.3\times10^6\,\text{mm}^4$$

Sample Problem 9.5

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

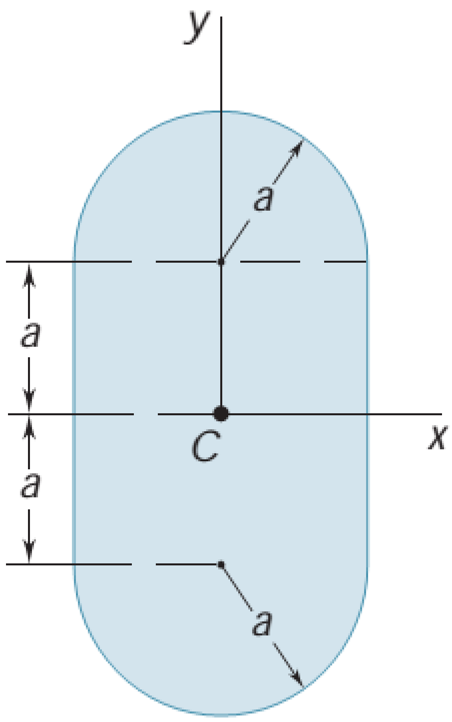


$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

Prob # 9.36

Determine the moments of inertia of the shaded area shown with respect to the x and y axes when a =20 mm.

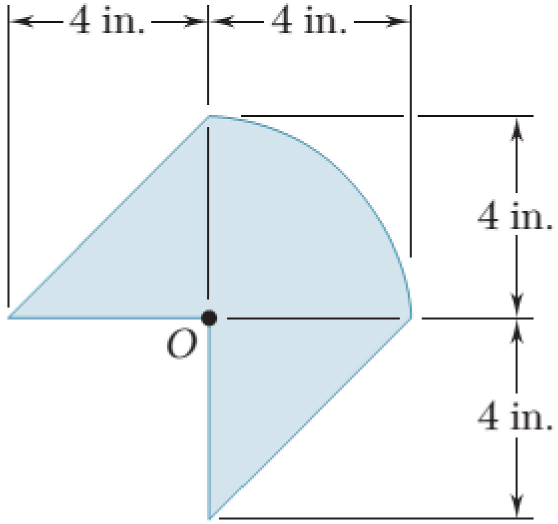


Prob # 9.46

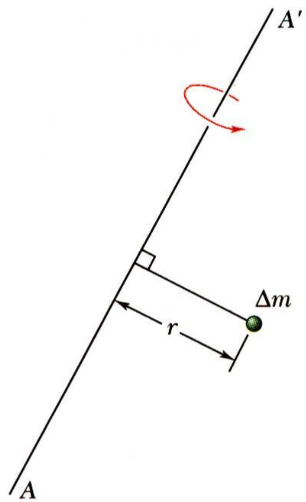
Determine the polar moment of inertia of the area shown with respect to

(a) point O,

(b) the centroid of the area.

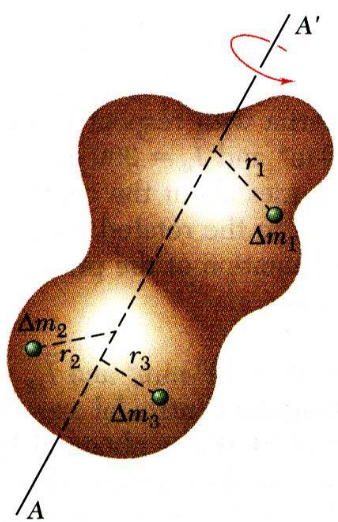


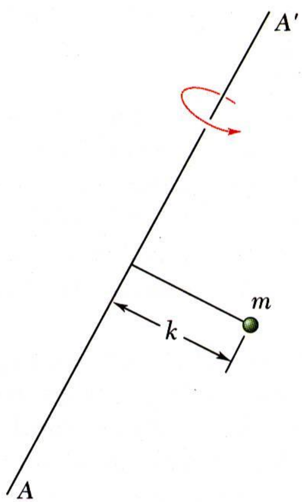
Moment of Inertia of a Mass



- Angular acceleration about the axis *AA'* of the small mass Δm due to the application of a couple is proportional to $r^2 \Delta m$.

  $r^2 \Delta m$ = *moment of inertia* of the mass Δm with respect to the axis *AA'*





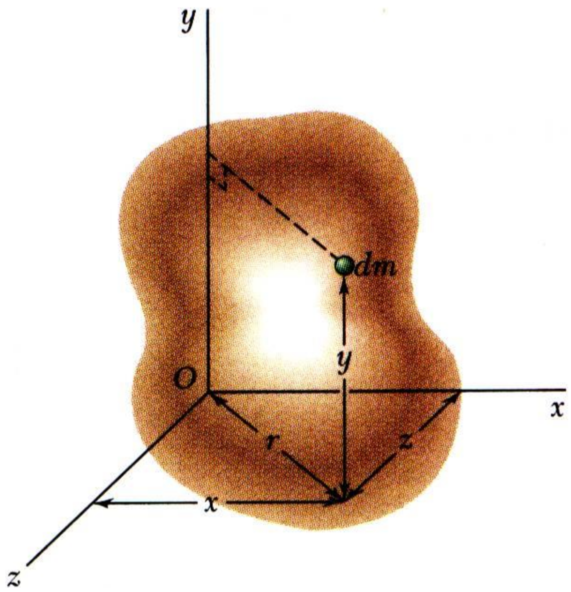
- For a body of mass m the resistance to rotation about the axis *AA' is*

$$I = r_1^2 \Delta m + r_2^2 \Delta m + r_3^2 \Delta m + \cdots$$
$$= \int r^2 dm = mass\ moment\ of\ inertia$$

- The radius of gyration for a concentrated mass with equivalent mass moment of inertia is

$$I = k^2 m \qquad k = \sqrt{\frac{I}{m}}$$

Moment of Inertia of a Mass



- Moment of inertia with respect to the *y* coordinate axis is

$$I_y = \int r^2 dm = \int \left(z^2 + x^2\right) dm$$

- Similarly, for the moment of inertia with respect to the *x* and *z* axes,

$$I_x = \int \left(y^2 + z^2\right) dm$$

$$I_z = \int \left(x^2 + y^2\right) dm$$

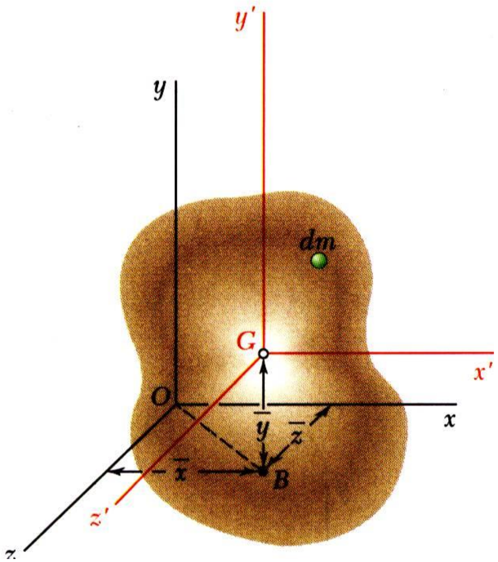
- In SI units,

$$I = \int r^2 dm = \left(\mathrm{kg} \cdot \mathrm{m}^2\right)$$

In U.S. customary units,

$$I = \left(slug \cdot ft^2\right) = \left(\frac{lb \cdot s^2}{ft} ft^2\right) = \left(lb \cdot ft \cdot s^2\right)$$

Parallel Axis Theorem



- For the rectangular axes with origin at *O* and parallel centroidal axes,

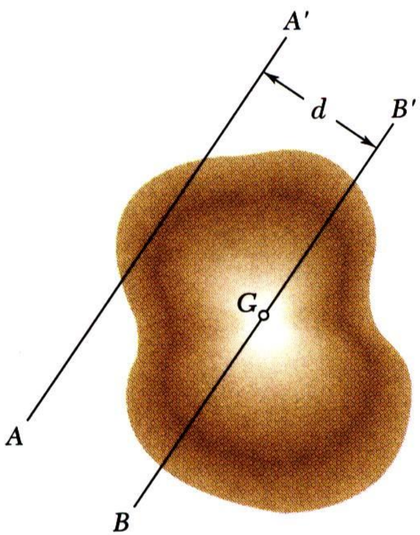
$$I_x = \int \left(y^2 + z^2\right) dm = \int \left[(y' + \bar{y})^2 + (z' + \bar{z})^2\right] dm$$

$$= \int \left(y'^2 + z'^2\right) dm + 2\bar{y} \int y' dm + 2\bar{z} \int z' dm + \left(\bar{y}^2 + \bar{z}^2\right) \int dm$$

$$I_x = \bar{I}_{x'} + m\left(\bar{y}^2 + \bar{z}^2\right)$$

$$I_y = \bar{I}_{y'} + m\left(\bar{z}^2 + \bar{x}^2\right)$$

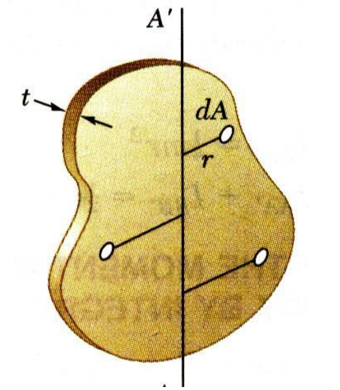
$$I_z = \bar{I}_{z'} + m\left(\bar{x}^2 + \bar{y}^2\right)$$



- Generalizing for any axis *AA'* and a parallel centroidal axis,

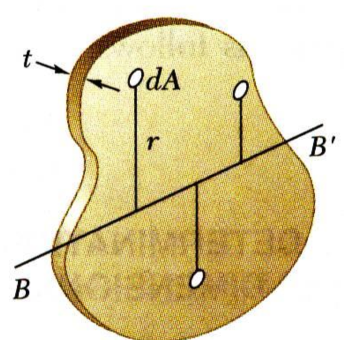
$$\boxed{I = \bar{I} + md^2}$$

Moments of Inertia of Thin Plates



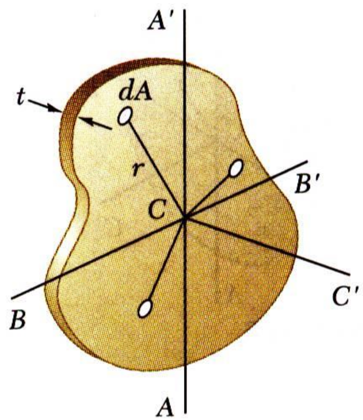
- For a thin plate of uniform thickness t and homogeneous material of density ρ, the mass moment of inertia with respect to axis AA' contained in the plate is

$$I_{AA'} = \int r^2 dm = \rho t \int r^2 dA$$
$$= \rho t \, I_{AA',area}$$



- Similarly, for perpendicular axis BB' which is also contained in the plate,

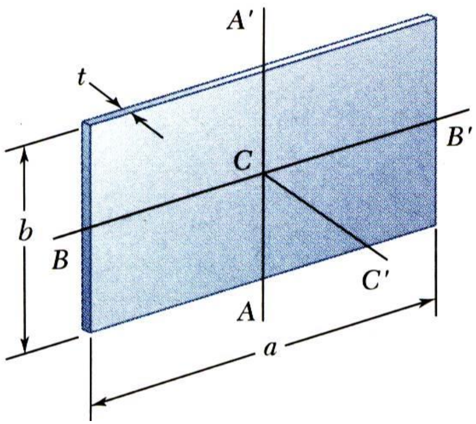
$$I_{BB'} = \rho t \, I_{BB',area}$$



- For the axis CC' which is perpendicular to the plate,

$$I_{CC'} = \rho t \, J_{C,area} = \rho t \left(I_{AA',area} + I_{BB',area} \right)$$
$$= I_{AA'} + I_{BB'}$$

Moments of Inertia of Thin Plates

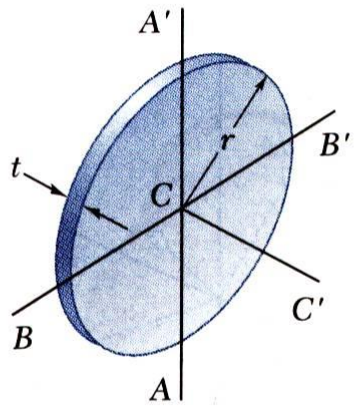


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{12}a^3 b\right) = \tfrac{1}{12}ma^2$$

$$I_{BB'} = \rho t\, I_{BB',area} = \rho t\left(\tfrac{1}{12}ab^3\right) = \tfrac{1}{12}mb^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12}m\left(a^2 + b^2\right)$$

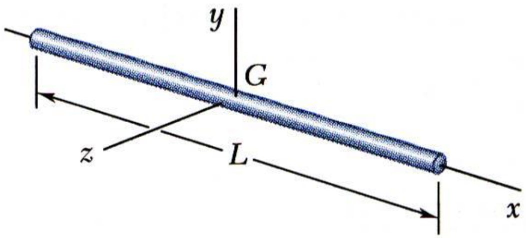


- For centroidal axes on a circular plate,

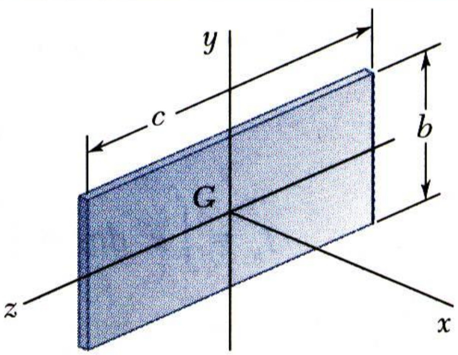
$$I_{AA'} = I_{BB'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{4}\pi r^4\right) = \tfrac{1}{4}mr^2$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2}mr^2$$

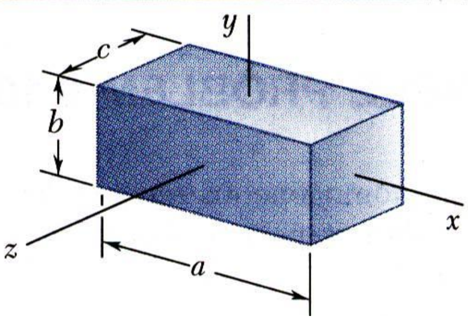
Moments of Inertia of Common Geometric Shapes



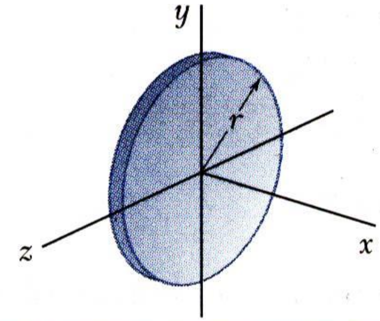
$$I_y = I_z = \frac{1}{12} mL^2$$



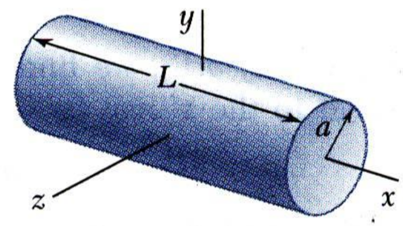
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} mc^2$$
$$I_z = \frac{1}{12} mb^2$$



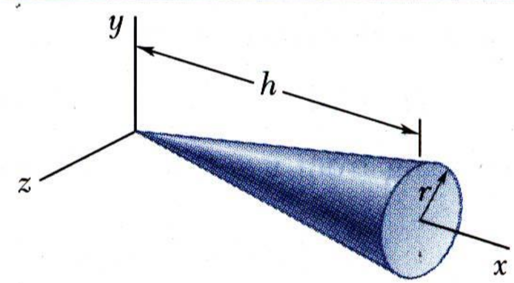
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} m(c^2 + a^2)$$
$$I_z = \frac{1}{12} m(a^2 + b^2)$$



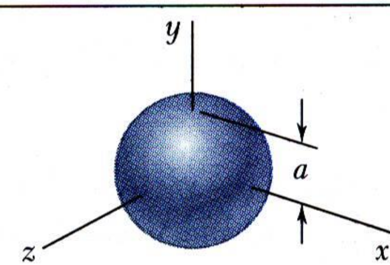
$$I_x = \frac{1}{2} mr^2$$
$$I_y = I_z = \frac{1}{4} mr^2$$



$$I_x = \frac{1}{2} ma^2$$
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$

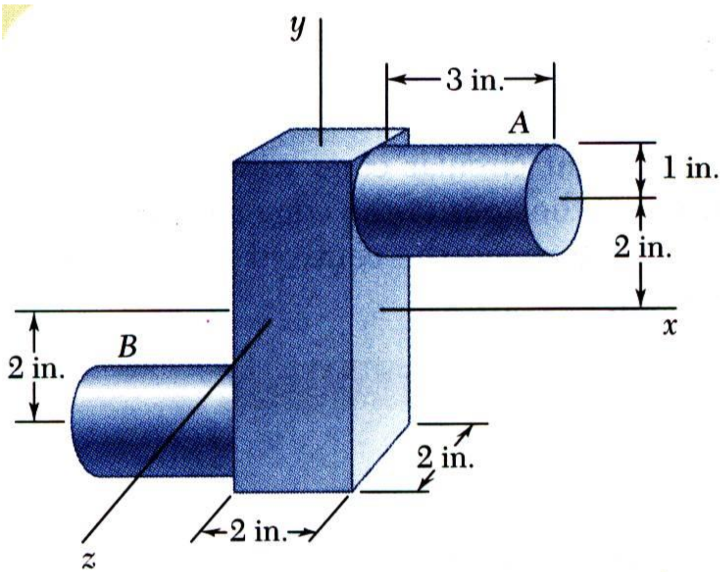


$$I_x = \frac{3}{10} ma^2$$
$$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$$



$$I_x = I_y = I_z = \frac{2}{5} ma^2$$

Sample Problem 9.12



Determine the moments of inertia of the steel forging with respect to the *xyz* coordinate axes, knowing that the specific weight of steel is 490 lb/ft3.

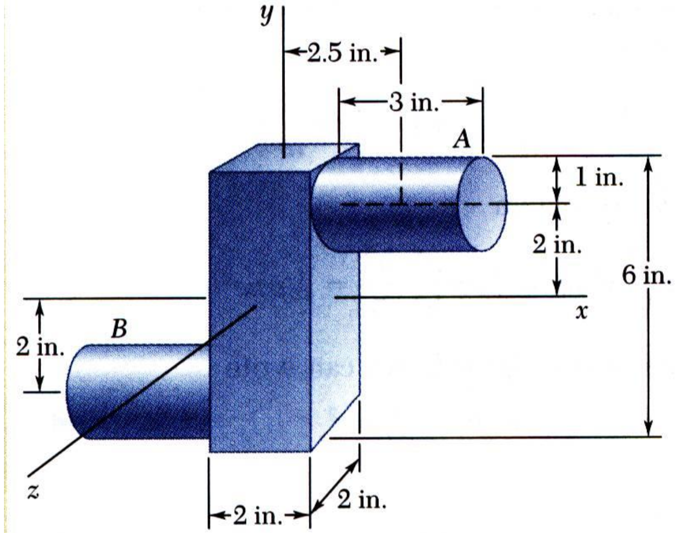
<u>SOLUTION</u>:

- With the forging divided into a prism and two cylinders, compute the mass and moments of inertia of each component with respect to the *xyz* axes using the parallel axis theorem.
- Add the moments of inertia from the components to determine the total moments of inertia for the forging.

Sample Problem 9.12

SOLUTION:

- Compute the moments of inertia of each component with respect to the *xyz* axes.



each cylinder :

$$m = \frac{\gamma V}{g} = \frac{\left(490\,\text{lb/ft}^3\right)\left(\pi \times 1^2 \times 3\right)\text{in}^3}{\left(1728\,\text{in}^3/\text{ft}^3\right)\left(32.2\,\text{ft}/\text{s}^2\right)}$$

$$m = 0.0829\ \text{lb}\cdot\text{s}^2/\text{ft}$$

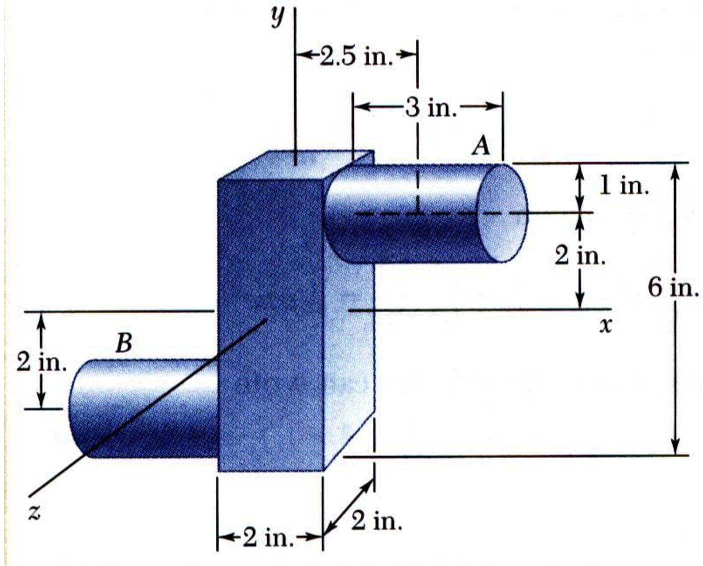
cylinders $\left(a = 1\text{in.}, L = 3\text{in.}, \bar{x} = 2.5\text{in.}, \bar{y} = 2\text{in.}\right)$:

$$\begin{aligned} I_x &= \tfrac{1}{2}ma^2 + m\bar{y}^2 \\ &= \tfrac{1}{2}(0.0829)\left(\tfrac{1}{12}\right)^2 + (0.0829)\left(\tfrac{2}{12}\right)^2 \\ &= 2.59\times10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

$$\begin{aligned} I_y &= \tfrac{1}{12}m\left[3a^2 + L^2\right] + m\bar{x}^2 \\ &= \tfrac{1}{12}(0.0829)\left[3\left(\tfrac{1}{12}\right)^2 + \left(\tfrac{3}{12}\right)^2\right] + (0.0829)\left(\tfrac{2.5}{12}\right)^2 \\ &= 4.17\times10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

$$\begin{aligned} I_z &= \tfrac{1}{12}m\left[3a^2 + L^2\right] + m\left[\bar{x}^2 + \bar{y}^2\right] \\ &= \tfrac{1}{12}(0.0829)\left[3\left(\tfrac{1}{12}\right)^2 + \left(\tfrac{3}{12}\right)^2\right] + (0.0829)\left[\left(\tfrac{2.5}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right] \\ &= 6.48\times10^{-3}\,\text{lb}\cdot\text{ft}\cdot\text{s}^2 \end{aligned}$$

Sample Problem 9.12



prism :

$$m = \frac{\gamma V}{g} = \frac{\left(490\ \text{lb/ft}^3\right)\left(2\times 2\times 6\right)\text{in}^3}{\left(1728\ \text{in}^3/\text{ft}^3\right)\left(32.2\ \text{ft/s}^2\right)}$$

$$m = 0.211\ \text{lb}\cdot\text{s}^2/\text{ft}$$

prism (a = 2 in., b = 6 in., c = 2 in.):

$$I_x = I_z = \tfrac{1}{12}m\left[b^2 + c^2\right] = \tfrac{1}{12}(0.211)\left[\left(\tfrac{6}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right]$$

$$= 4.88\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = \tfrac{1}{12}m\left[c^2 + a^2\right] = \tfrac{1}{12}(0.211)\left[\left(\tfrac{2}{12}\right)^2 + \left(\tfrac{2}{12}\right)^2\right]$$

$$= 0.977\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

- Add the moments of inertia from the components to determine the total moments of inertia.

$$I_x = 4.88\times 10^{-3} + 2\left(2.59\times 10^{-3}\right)$$

$$\boxed{I_x = 10.06\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

$$I_y = 0.977\times 10^{-3} + 2\left(4.17\times 10^{-3}\right)$$

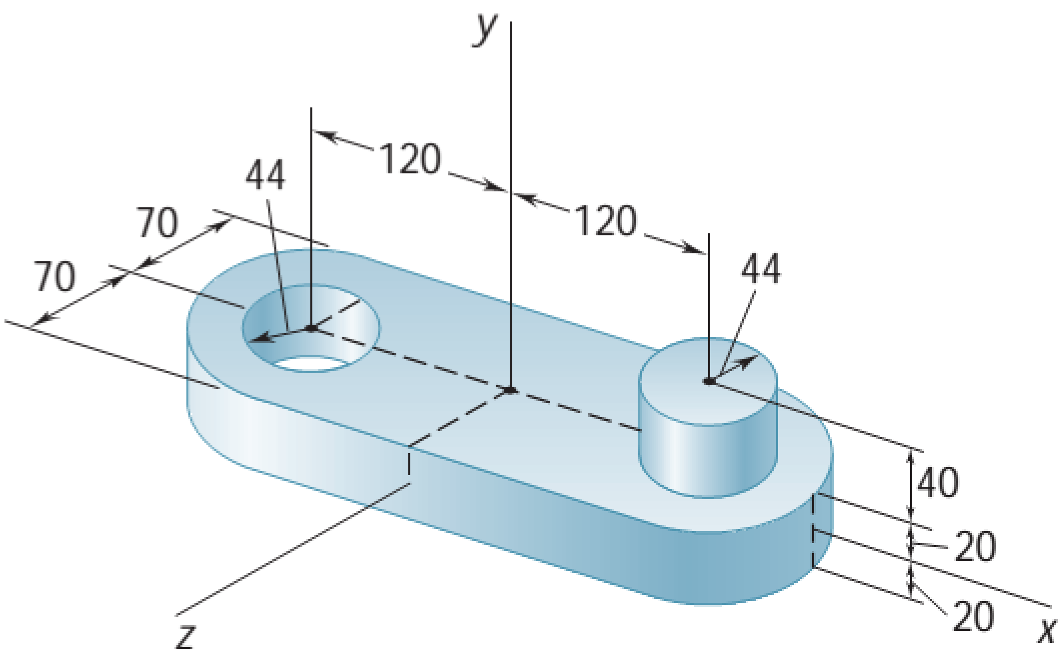
$$\boxed{I_y = 9.32\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

$$I_z = 4.88\times 10^{-3} + 2\left(6.48\times 10^{-3}\right)$$

$$\boxed{I_z = 17.84\times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2}$$

Prob # 9.142

Determine the mass moments of inertia and the radii of gyration of the steel machine element shown with respect to y axis. (The density of steel is 7850 kg/m^3)



Dimensions in mm